CONSISTENCY OF KERNEL-TYPE ESTIMATORS FOR
THE FIRST AND SECOND DERIVATIVES OF A
PERIODIC POISSON INTENSITY FUNCTION

I WAYAN MANGKU, SYAMSURI, HERNIWATI

Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Bogor Agricultural University
Jl. Meranti, Kampus IPB Darmaga, Bogor, 16680 Indonesia

Abstract. We construct and investigate consistent kernel-type
estimators for the first and second derivatives of a periodic Pois-
on intensity function when the period is known. We do not ass ume
any particular parametric form for the intensity function. More-
over, we consider the situation when only a single realization of the
Poisson process is available, and only observed in a bounded inter-
val. We prove that the proposed estimators are consistent when
the length of the interval goes to infinity. We also prove that the
mean-squared error of the estimators converge to zero when t he
length of the interval goes to infinity.


Keywords and Phrases: periodic Poisson process, intensity func-
tion, kernel type estimator, consistency, first derivative, second
derivative.

1. Introduction

We consider kernel type estimations for the first and second deriv-
atives of the intensity function of a periodic Poisson process. Let $N$ be
a Poisson process on $[0, \infty)$ with (unknown) locally integrable inten-
sity function $\lambda$. We assume that $\lambda$ is a periodic function with (known)
period $\tau$. We do not assume any parametric form of $\lambda$, except that it
is periodic. That is, for each point $s \in [0, \infty)$ and all $k \in \mathbb{Z}$, with $\mathbb{Z}$
denotes the set of integers, we have

$$\lambda(s + k\tau) = \lambda(s). \quad (1.1)$$

Suppose that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the Pois-
on process $N$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with intensity
function $\lambda$ is observed, though only within a bounded interval $[0, n]$.
Our goal in this paper is to study consistency of estimators for the
first and second derivatives of the intensity function $\lambda$ at a given point
\(s \in [0, \infty)\) using only a single realization \(N(\omega)\) of the Poisson process \(N\) observed in interval \([0, n]\). A special case study using uniform kernel estimators can be found in [4].

Throughout this paper, we assume that \(s\) is a Lebesgue point of \(\lambda\), that is we have
\[
\lim_{h \downarrow 0} \frac{1}{2h} \int_{-h}^{h} |\lambda(s + x) - \lambda(s)|\,dx = 0 \quad (1.2)
\]
(eg. see [7], p.107-108).

Since \(\lambda\) is a periodic function with period \(\tau\), the problem of estimating \(\lambda, \lambda'\) (the first derivative of \(\lambda\)) and \(\lambda''\) (the second derivative of \(\lambda\)) at a given point \(s \in [0, \infty)\) can be reduced into a problem of estimating \(\lambda, \lambda'\) and \(\lambda''\) at a given point \(s \in [0, \tau]\). Hence, for the rest of this paper, we assume that \(s \in [0, \tau]\).

2. The estimators and some results

To define estimators of \(\lambda'\) and \(\lambda''\) we need an estimator of \(\lambda\). Therefore, before defining estimators of \(\lambda'\) and \(\lambda''\), we first review the construction of a kernel-type estimator of \(\lambda\) at a given point \(s\), as given in Helmers et al. [2], as follows. Let \(K : \mathbb{R} \rightarrow \mathbb{R}\) be a real valued function, called kernel, which satisfies the following conditions: (K1) \(K\) is a probability density function, (K2) \(K\) is bounded, and (K3) \(K\) has (closed) support \([-1, 1]\). Let also \(h_n\) be a sequence of positive real numbers converging to 0, that is,
\[
h_n \downarrow 0, \quad (2.1)
\]
as \(n \rightarrow \infty\). Now, we may define the estimator of \(\lambda\) at a given point \(s \in [0, \tau]\) as follows
\[
\hat{\lambda}_{n,K}(s) := \frac{\tau}{n} \sum_{k=0}^{n-1} \frac{1}{h_n} \int_{0}^{h_n} K \left( \frac{x - (s + k\tau)}{h_n} \right) N(dx), \quad (2.2)
\]
This estimator is a special case of a more general kernel-type estimator of the intensity of a periodic Poisson process, which includes the case when the period \(\tau\) has to be estimated (see Helmers et al. ([2], [3])).

By having the estimator of \(\lambda\) at a given point \(s \in [0, \tau]\), following the idea in Helmers and Mangku [1], we may define an estimator of \(\lambda'\) at a given point \(s \in [0, \tau]\) as follows
\[
\hat{\lambda}'_{n,K}(s) := \frac{\hat{\lambda}_{n,K}(s + h_n) - \hat{\lambda}_{n,K}(s - h_n)}{2h_n}. \quad (2.3)
\]
Construction of this estimator is using the fact that, for small \(h\) we have
\[
\lambda'(s) \approx \frac{\lambda(s + h) - \lambda(s - h)}{2h}.
\]
Consistency of \(\hat{\lambda}'_{n,K}(s)\) is given in the following theorem.
Theorem 2.1. (Consistency of $\hat{\lambda}'_{n,K}(s)$)
Suppose that the intensity function $\lambda$ is periodic and locally integrable, and has finite first derivative $\lambda'$ at $s$. If the kernel $K$ is symmetric and satisfies conditions $(K1), (K2), (K3)$, and $h_n$ satisfies assumptions (2.1) and $nh_n^3 \to \infty$, then

$$\hat{\lambda}'_{n,K}(s) \overset{p}{\to} \lambda'(s),$$

as $n \to \infty$. In other words, $\hat{\lambda}'_{n,K}(s)$ is a consistent estimator of $\lambda'(s)$. In addition, the mean-squared error (MSE) of $\hat{\lambda}'_{n,K}(s)$ converges to 0, as $n \to \infty$.

Next we consider estimation of the second derivative $\lambda''$ of $\lambda$ at a given point $s \in [0, \tau)$. Following the idea in Helmers and Mangku [1], we may define an estimator of $\lambda''$ at a given point $s \in [0, \tau)$ as follows

$$\hat{\lambda}''_{n,K}(s) := \frac{\hat{\lambda}_{n,K}(s + 2h_n) + \hat{\lambda}_{n,K}(s - 2h_n) - 2\hat{\lambda}_{n,K}(s)}{4h_n^2}. \quad (2.5)$$

Construction of this estimator is using the fact that, for small $h$ we have

$$\lambda''(s) \approx \frac{\lambda'(s + h) - \lambda'(s - h)}{2h} \approx \frac{\lambda(s + 2h) + \lambda(s - 2h) - 2\lambda(s)}{4h^2}.$$

Consistency of $\hat{\lambda}''_{n,K}(s)$ is given in the following theorem.

Theorem 2.2. (Consistency of $\hat{\lambda}''_{n,K}(s)$)
Suppose that the intensity function $\lambda$ is periodic and locally integrable, and has finite second derivative $\lambda''$ at $s$. If the kernel $K$ is symmetric and satisfies conditions $(K1), (K2), (K3)$, and $h_n$ satisfies assumptions (2.1) and $nh_n^5 \to \infty$, then

$$\hat{\lambda}''_{n,K}(s) \overset{p}{\to} \lambda''(s),$$

as $n \to \infty$. In other words, $\hat{\lambda}''_{n,K}(s)$ is a consistent estimator of $\lambda''(s)$. In addition, the MSE of $\hat{\lambda}''_{n,K}(s)$ converges to 0, as $n \to \infty$.

3. Some Technical Lemmas
To prove Theorems 2.1 and 2.2 we need the following two lemmas. The first lemma is about asymptotic approximations to $E\lambda_{n,K}(s)$ in two cases, namely (i) when $\lambda$ has finite first derivative at $s$, (ii) when $\lambda$ has finite second derivative at $s$. The second lemma is about asymptotic approximation to the variance of $\hat{\lambda}_{n,K}(s)$. We will use the first lemma to prove that the bias of $\hat{\lambda}'_{n,K}(s)$ and $\hat{\lambda}''_{n,K}(s)$ converge to zero as $n \to \infty$. 
The second lemma will be used to prove that the variances of $\hat{\lambda}_n(s)$ and $\hat{\lambda}''_n(s)$ converge to zero as $n \to \infty$.

**Lemma 3.1. (Asymptotic approximations to the bias of $\hat{\lambda}_n(s)$)**

Suppose that the intensity function $\lambda$ is periodic and locally integrable, the kernel $K$ is symmetric and satisfies conditions (K1), (K2), (K3), and $h_n$ satisfies assumptions (2.1).

(i) If $nh_n \to \infty$ and $\lambda$ has finite first derivative at $s$ then

$$E\hat{\lambda}_n(s) = \lambda(s) + o(h_n),$$

as $n \to \infty$.

(ii) If $nh_n^2 \to \infty$ and $\lambda$ has finite second derivative at $s$ then

$$E\hat{\lambda}_n(s) = \lambda(s) + \frac{1}{2} \lambda''(s)h_n^2 \int_{-1}^{1} x^2 K(x)dx + o(h_n^2),$$

as $n \to \infty$.

**Proof:** Here we only give the proof of part (i) of this lemma (see also [6]). Proof of part (ii) of this lemma can be found in [5]. To prove (3.1), first note that

$$E\hat{\lambda}_n(s) = \frac{\tau}{n} \sum_{k=0}^{\infty} \int_{0}^{n} K \left( \frac{x - (s + k\tau)}{h_n} \right) \lambda(x)dx$$

$$= \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n} \int_{0}^{n} K \left( \frac{x - (s + k\tau)}{h_n} \right) \lambda(x)dx$$

$$= \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n} \int_{\mathbb{R}} K \left( \frac{x}{h_n} \right) \lambda(x)I(x \in [0, n])dx. \quad (3.3)$$

By a change of variable and using (1.1), we can write the r.h.s. of (3.3) as

$$\frac{\tau}{n} \sum_{k=0}^{\infty} \int_{\mathbb{R}} K \left( \frac{x}{h_n} \right) \lambda(x + s + k\tau)I(x + s + k\tau \in [0, n])dx$$

$$= \frac{\tau}{nh_n} \int_{\mathbb{R}} K \left( \frac{x}{h_n} \right) \lambda(x + s)I(x + s \in [0, n])dx$$

$$= \frac{\tau}{nh_n} \int_{\mathbb{R}} K \left( \frac{x}{h_n} \right) \lambda(x + s) \sum_{k=0}^{\infty} I(x + s + k\tau \in [0, n])dx. \quad (3.4)$$

Now note that

$$\sum_{k=0}^{\infty} I(x + s + k\tau \in [0, n]) \in \left[ \frac{n}{\tau} - 1, \frac{n}{\tau} - 1 \right]. \quad (3.5)$$
Then, the r.h.s. of (3.4) can be written as
\[
\frac{\tau}{n h_n} \int_R K \left( \frac{x}{h_n} \right) \lambda(x + s) \left( \frac{n}{\tau} + O(1) \right) dx
\]
\[
= \frac{1}{h_n} \int_R K \left( \frac{x}{h_n} \right) \lambda(x + s) dx + O \left( \frac{1}{n} \right)
\]
\[
= \int_{-1}^{1} K(x) \lambda(s + x h_n) dx + O \left( \frac{1}{n} \right),
\]
(3.6)
as \( n \to \infty \). By the Young’s form of Taylor’s theorem, we have
\[
\lambda(s + x h_n) = \lambda(s) + \frac{\lambda'(s)}{1!} x h_n + o(h_n),
\]
(3.7)
jika \( n \to \infty \). Substituting (3.7) into the r.h.s. of (3.6), we obtain
\[
E \hat{\lambda}_{n,K}(s) = \int_{-1}^{1} K(x) \left( \lambda(s) + \frac{\lambda'(s)}{1!} x h_n \right) dx + o(h_n) + O \left( \frac{1}{n} \right)
\]
\[
= \lambda(s) \int_{-1}^{1} K(x) dx + \lambda'(s) h_n \int_{-1}^{1} x K(x) dx + o(h_n)
\]
\[
+ O \left( \frac{1}{n} \right),
\]
(3.8)
as \( n \to \infty \). By assumption (K1) and (K3) we have \( \int_{-1}^{1} K(x) dx = 1 \).
Since the kernel \( K \) is symmetric, an easy calculation shows that the second term on the r.h.s. of (3.8) is equal to zero. By the assumption \( nh_n \to \infty \), we have the last term on the r.h.s. of (3.8) is of order \( o(h_n) \), as \( n \to \infty \). Hence we obtain (3.1). This completes the proof of part (i) of Lemma 3.1.

**Lemma 3.2. (Asymptotic approximation to the variance of \( \hat{\lambda}_{n,K}(s) \))**

Suppose that the intensity function \( \lambda \) is periodic and locally integrable. If the kernel \( K \) satisfies conditions \((K1),(K2),(K3)\), and \( h_n \) satisfies assumptions \((2.1)\), then
\[
Var \left( \hat{\lambda}_{n,K}(s) \right) = \frac{\tau \lambda(s)}{n h_n} \int_{-1}^{1} K^2(x) dx + o \left( \frac{1}{nh_n} \right)
\]
(3.9)
as \( n \to \infty \), provided \( s \) is a Lebesgue point of \( \lambda \).

**Proof:** We refer to [5] for the proof of this lemma.

**4. Proof of Theorem 2.1**

To prove Theorem 2.1 it suffices to check the following two lemmas.

**Lemma 4.1. (Asymptotic unbiasedness of \( \hat{\lambda}_{n,K}'(s) \))**

Suppose that the intensity function \( \lambda \) is periodic, locally integrable and has finite first derivative at \( s \). If the kernel \( K \) is symmetric and satisfies
conditions \( (K1), (K2), (K3) \), the bandwidth \( h_n \) satisfies assumptions (2.1) and \( nh_n \to \infty \), then

\[
\mathbb{E}\hat{\lambda}'_{n,K}(s) \to \lambda'(s),
\]

as \( n \to \infty \). In other words, \( \hat{\lambda}'_{n,K}(s) \) is asymptotically unbiased estimator of \( \lambda'(s) \).

Proof: By (2.3), \( \mathbb{E}\hat{\lambda}'_{n,K}(s) \) can be computed as follows

\[
\mathbb{E}\hat{\lambda}'_{n,K}(s) = \frac{1}{2h_n} \left( \mathbb{E}\hat{\lambda}_{n,K}(s + h_n) - \mathbb{E}\hat{\lambda}_{n,K}(s - h_n) \right).
\]

By (3.1) and Taylor expansion we have

\[
\mathbb{E}\hat{\lambda}_{n,K}(s + h_n) = \lambda(s + h_n) + o(h_n) = \lambda(s) + \frac{\lambda'(s)}{1!} h_n + o(h_n)
\]

and

\[
\mathbb{E}\hat{\lambda}_{n,K}(s - h_n) = \lambda(s - h_n) + o(h_n) = \lambda(s) - \frac{\lambda'(s)}{1!} h_n + o(h_n)
\]

as \( n \to \infty \). Substituting (4.3) and (4.4) into the r.h.s. of (4.2), then we obtain

\[
\mathbb{E}\hat{\lambda}'_{n,K}(s) = \lambda'(s) + o(1)
\]

as \( n \to \infty \), which is equivalent to (4.1). This completes the proof of Lemma 4.1.

Lemma 4.2. (Convergency of the variance of \( \hat{\lambda}'_{n,K}(s) \))

Suppose that the intensity function \( \lambda \) is periodic and locally integrable. If the kernel \( K \) satisfies conditions \( (K1), (K2), (K3) \), and \( h_n \) satisfies assumptions (2.1) and \( nh_n^2 \to \infty \), then

\[
\text{Var} \left( \hat{\lambda}'_{n,K}(s) \right) \to 0,
\]

as \( n \to \infty \), provided \( s \) is a Lebesgue point of \( \lambda \).

Proof: By (2.3), \( \text{Var}(\hat{\lambda}'_{n,K}(s)) \) can be computed as follows

\[
\text{Var} \left( \hat{\lambda}'_{n,K}(s) \right) = \frac{1}{4h_n^2} \left( \text{Var}(\hat{\lambda}_{n,K}(s + h_n)) + \text{Var}(\hat{\lambda}_{n,K}(s - h_n)) \\
-2\text{Cov}(\hat{\lambda}_{n,K}(s + h_n), \hat{\lambda}_{n,K}(s - h_n)) \right).
\]

By (2.1), for sufficiently large \( n \), we have that for each integer \( k \), the interval \([s + k\tau, s + k\tau + 2hn]\) and \([s + k\tau - 2hn, s + k\tau]\) are disjoint. This means that \( \hat{\lambda}_{n,K}(s + h_n) \) and \( \hat{\lambda}_{n,K}(s - h_n) \) are independent, which implies \( \text{Cov}(\hat{\lambda}_{n,K}(s + h_n), \hat{\lambda}_{n,K}(s - h_n)) = 0 \). Then (4.6) reduces to

\[
\text{Var} \left( \hat{\lambda}'_{n,K}(s) \right) = \frac{1}{4h_n^2} \left( \text{Var}(\hat{\lambda}_{n,K}(s + h_n)) + \text{Var}(\hat{\lambda}_{n,K}(s - h_n)) \right),
\]

(4.7)
By (3.9) we obtain that
\[
\text{Var} \left( \hat{\lambda}'_{n,K}(s) \right) = \frac{1}{4h_n^2} O \left( \frac{1}{nh_n} \right) = O \left( \frac{1}{nh_n^3} \right),
\]
as \( n \to \infty \). By the assumption \( nh_n^3 \to \infty \), we obtain (4.5). This completes the proof of Lemma 4.2.

5. PROOF OF THEOREM 2.2

To prove Theorem 2.2 it suffices to check the following two lemmas.

Lemma 5.1. (Asymptotic unbiasedness of \( \hat{\lambda}''_{n,K}(s) \))

Suppose that the intensity function \( \lambda \) is periodic, locally integrable and has finite second derivative at \( s \). If the kernel \( K \) is symmetric and satisfies conditions \((K1),(K2),(K3)\), the bandwidth \( h_n \) satisfies assumptions (2.1) and \( nh_n^2 \to \infty \), then
\[
E \hat{\lambda}''_{n,K}(s) \to \lambda''(s),
\]
as \( n \to \infty \). In other words, \( \hat{\lambda}''_{n,K}(s) \) is asymptotically unbiased estimator of \( \lambda''(s) \).

Proof: By (2.5), \( E \hat{\lambda}''_{n,K}(s) \) can be computed as follows
\[
E \hat{\lambda}''_{n,K}(s) = \frac{1}{4h_n^2} \left( E \hat{\lambda}_{n,K}(s+2h_n) + E \hat{\lambda}_{n,K}(s-2h_n) - 2E \hat{\lambda}_{n,K}(s) \right).
\] (5.2)

By (3.2) we have
\[
E \hat{\lambda}_{n,K}(s+2h_n) = \lambda(s+2h_n) + \frac{1}{2} \lambda''(s+2h_n)h_n^2 \int_{-1}^{1} x^2 K(x) dx + o(h_n^2),
\] (5.3)
and
\[
E \hat{\lambda}_{n,K}(s-2h_n) = \lambda(s-2h_n) + \frac{1}{2} \lambda''(s-2h_n)h_n^2 \int_{-1}^{1} x^2 K(x) dx + o(h_n^2),
\] (5.4)
as \( n \to \infty \). By Taylor expansion we obtain
\[
\lambda(s+2h_n) = \lambda(s) + \frac{\lambda'(s)}{1!} 2h_n + \frac{\lambda''(s)}{2!} 4h_n^2 + o(h_n^2),
\] (5.5)
\[
\lambda(s-2h_n) = \lambda(s) - \frac{\lambda'(s)}{1!} 2h_n + \frac{\lambda''(s)}{2!} 4h_n^2 + o(h_n^2),
\] (5.6)
\[
\lambda''(s+2h_n) = \lambda''(s) + o(1),
\] (5.7)
\[
\lambda''(s-2h_n) = \lambda''(s) + o(1),
\] (5.8)
as $n \to \infty$. Substituting (5.5) and (5.7) into the r.h.s. of (5.3), we obtain
\[
E\hat{\lambda}_{n,K}(s + 2h_n) = \lambda(s) + \frac{\lambda'(s)}{1!}2h_n + \frac{\lambda''(s)}{2!}4h_n^2 \\
+ \frac{1}{2}\lambda''(s)h_n^2\int_{-1}^{1}x^2K(x)dx + o(h_n^2), \quad (5.9)
\]
as $n \to \infty$. Substituting (5.6) and (5.8) into the r.h.s. of (5.4), we obtain
\[
E\hat{\lambda}_{n,K}(s - 2h_n) = \lambda(s) - \frac{\lambda'(s)}{1!}2h_n + \frac{\lambda''(s)}{2!}4h_n^2 \\
+ \frac{1}{2}\lambda''(s)h_n^2\int_{-1}^{1}x^2K(x)dx + o(h_n^2), \quad (5.10)
\]
as $n \to \infty$. Finally, by substituting (3.2), (5.9) and (5.10) into the r.h.s. of (5.2), we obtain
\[
E\hat{\lambda}_{n,K}(s) = \lambda''(s) + o(1)
\]
as $n \to \infty$, which is equivalent to (5.1). This completes the proof of Lemma 5.1.

**Lemma 5.2. (Convergence of the variance of $\hat{\lambda}_{n,K}'(s)$)**

Suppose that the intensity function $\lambda$ is periodic and locally integrable. If the kernel $K$ satisfies conditions (K1), (K2), (K3), and $h_n$ satisfies assumptions (2.1) and $nh_n^5 \to \infty$, then
\[
Var\left(\hat{\lambda}_{n,K}'(s)\right) \to 0, \quad (5.11)
\]
as $n \to \infty$, provided $s$ is a Lebesgue point of $\lambda$.

**Proof:** By a simple argument, see proof of Lemma 4.2, for sufficiently large $n$ we have that $\lambda_{n,K}(s + 2h_n)$, $\lambda_{n,K}(s - 2h_n)$ and $\hat{\lambda}_{n,K}(s)$ are independent. Then, by (2.5), $Var(\hat{\lambda}_{n,K}'(s))$ can be computed as follows
\[
Var\left(\hat{\lambda}_{n,K}'(s)\right) = \frac{1}{16h_n^4}\left(Var(\hat{\lambda}_{n,K}(s + 2h_n)) + Var(\hat{\lambda}_{n,K}(s - 2h_n))
+ 4Var(\hat{\lambda}_{n,K}(s))\right). \quad (5.12)
\]
By (3.9) we obtain that
\[
Var\left(\hat{\lambda}_{n,K}'(s)\right) = \frac{1}{16h_n^4}\mathcal{O}\left(\frac{1}{nh_n}\right) = \mathcal{O}\left(\frac{1}{nh_n^5}\right),
\]
as $n \to \infty$. By the assumption $nh_n^5 \to \infty$, we obtain (5.11). This completes the proof of Lemma 5.2.
References


