PERSATUAN AKTUARIS INDONESIA

UJIAN PROFESI AKTUARIS

MATA UJIAN : A70 - Permodelan dan Teori Risiko
TANGGAL : 29 November 2011
JAM : 13.30 – 16.30

LAMA UJIAN : 180 Menit
SIFAT UJIAN : Tutup Buku

2011

TATA TERTIB UJIAN

1. Setiap Kandidat harus berada di ruang ujian selambat-lambatnya 15 (lima belas) menit sebelum ujian dimulai.
2. Kandidat yang datang 1 (satu) jam setelah berlangsungnya ujian dilarang memasuki ruang ujian dan mengikuti ujian.
3. Kandidat dilarang meninggalkan ruang ujian selama 1 (satu) jam pertama berlangsungnya ujian.
4. Setiap kandidat harus menempati bangku yang telah ditentukan oleh Komisi Penguji.
5. Buku-buku, diktat, dan segala jenis catatan harus diletakkan di tempat yang sudah ditentukan oleh Pengawas, kecuali alat tulis yang diperlukan untuk mengerjakan ujian dan kalkulator.
6. Setiap kandidat hanya berhak memperoleh satu set bahan ujian. Kerusakan lembar jawaban oleh kandidat, tidak akan diganti. Dalam memberikan jawaban, lembar jawaban harus dijaga agar tidak kotor karena coretan.
7. Kandidat dilarang berbicara dengan/atau melihat pekerjaan kandidat lain atau berkomunikasi langsung ataupun tidak langsung dengan kandidat lainnya selama ujian berlangsung.
8. Kandidat dilarang menanyakan makna pertanyaan kepada Pengawas ujian.
9. Kandidat yang terpaksa harus meninggalkan ruang ujian untuk keperluan mendesak (misalnya ke toilet) harus meminta izin kepada Pengawas ujian dan setiap kali izin keluar diberikan hanya untuk 1 (satu) orang.
10. Alat komunikasi (telepon seluler, pager, dan lain-lain) harus dimatikan selama ujian berlangsung.
11. Pengawas akan mencatat semua jenis pelanggaran atas tata tertib ujian yang akan menjadi pertimbangan diskualifikasi.
12. Kandidat yang telah selesai mengerjakan soal ujian, harus menyerahkan lembar jawaban langsung kepada Pengawas ujian dan tidak meninggalkan lembar jawaban tersebut di meja ujian.
13. Kandidat yang telah menyerahkan lembar jawaban harus meninggalkan ruang ujian.
14. Kandidat dapat mengajukan keberatan terhadap soal ujian yang dinilai tidak benar dengan penjelasan yang memadai kepada komisi penguji selambat-lambatnya 5 (lima) hari kerja sejak tanggal pelaksanaan ujian.

PETUNJUK MENGERJAKAN SOAL

Ujian Pilihan Ganda

1. Setiap soal akan mempunyai 4 (empat) pilihan jawaban di mana hanya 1 (satu) jawaban yang benar.
2. Setiap soal mempunyai bobot nilai yang sama dengan tidak ada pengurangan nilai untuk jawaban yang salah.
3. Berilah tanda silang pada jawaban yang Saudara anggap benar di lembar jawaban. Jika Saudara telah menentukan jawaban dan kemudian ingin merubahnya dengan yang lain, maka coretlah jawaban yang salah dan silang jawaban yang benar.
4. Jangan lupa **menuliskan nomor ujian Saudara pada** tempat yang sediakan dan **tanda tangani lembar jawaban tersebut tanpa menuliskan nama Saudara.**

Ujian Soal Esay

1. Setiap soal dapat mempunyai lebih dari 1 (satu) pertanyaan, Setiap soal mempunyai bobot yang sama kecuali terdapat keterangan pada soal.
2. Tuliskan jawaban Saudara pada Buku Jawaban Soal dengan jelas, rapi dan terstruktur sehingga akan mempermudah pemeriksaan hasil ujian.
3. Saudara bisa mulai dengan soal yang anda anggap mudah dan tuliskan nomor jawaban soal dengan soal dengan jelas.
4. Jangan lupa **menuliskan nomor ujian Saudara** pada tempat yang disediakan dan **tanda tangani Buku Ujian tanpa menuliskan nama Saudara.**

KETENTUAN DAN PROSEDUR KEBERATAN SOAL UJIAN PAI

1. Kandidat dapat mengajukan keberatan terhadap soal ujian yang dinilai tidak benar dengan penjelasan yang memadai kepada komisi penguji **selambat-lambatnya 5 (lima) hari kerja sejak tanggal pelaksanaan ujian.**
2. Semua pengajuan keberatan soal dialamatkan ke sanggahan.soal@gmail.com.
3. Pengajuan keberatan soal setelah tanggal tersebut (Poin No 1) tidak akan diterima dan ditanggapi.
4. Kunci Jawaban akan diumumkan ke website PAI dengan alamat www.aktuaris.org satu bulan sejak tanggal pelaksanaan ujian, setelah itu tidak ada pengajuan keberatan kunci jawaban.

1. X adalah sebuah distribusi gamma dengan mean = 8 dan skewness = 1. Tentukan variansi dari X
    a. 4
    b. 8
    c. 16
    d. 32

Data berikut untuk nomor 2 – 4.
Diketahui data berikut adalah besaran dari klaim kendaraan bermotor:
2, 5, 5, 7, 9, 12, 15, 18, 20, 25, 26, 35, 40, 50, 65

2. Berapakah unbiased sample variance nya?
    a. 313
    b. 315
    c. 335
    d. 350
3. Berapakah *p* value untuk statistic tes dari hipotesis yang menyatakan bahwa rata2 klaim tersebut sebesar 12
    a. 2%
    b. 3%
    c. 4%
    d. 5%
4. Berapakah batas atas dari 95% confidence interval?
    a. Lebih kecil dari 32
    b. Antara 32 dan 35
    c. Antara 35 dan 38
    d. Lebih besar dari 38

Data berikut diambil dari sebuah populasi untuk soal no 5 - 9
7, 10, 12, 15, 18, 25, 30, 35, 100, 150, 200, 275

5. Berapakah Kurtosis dari data tersebut?
    a. Kurang dari 2
    b. Antara 2 dan 2,5
    c. Antara 2,5 dan 3
    d. Lebih dari 3
6. Apabila θ parameter distribusi eksponensial dicari dengan metode Maximum Likelihood Estimation, berapakah nilainya?
    a. 73
    b. 74
    c. 75
    d. 76
7. Tentukan estimasi $P[X > 30]$ apabila distribusi tersebut adalah Inverse Exponential yang didekati dengan methode percentile 25%
    a. 0,46
    b. 0,48

c. 0,50
d. 0,52

8. Berapakah nilai $\alpha + \theta$ parameter-parameter dari distribusi gamma yang didekati dengan metode momen
   a. Kurang dari 100
   b. Antara 100 dan 101
   c. Antara 101 dan 102
   d. Lebih dari 102

9. Tentukan estimasi empiris (empirical estimates) dari variansi untuk $X \wedge 100$ atau variansi X dengan limit 100
   a. 25
   b. 46
   c. 73
   d. 100

Data berikut digunakan untuk nomor 10 – 12
Terdapat 12 data yang diambil dari sebuah populasi distribusi: 7 , 12, 15, 19, 26, 27, 29, 29, 30, 33, 38, 53, estimasi dari distribusi tersebut adalah eksponensial dengan estimasi parameter $\theta = 30$

10. Tentukan Kolmogorov Smirnov statistic untuk data dan model tersebut
    a. 0,19
    b. 0,21
    c. 0,23
    d. 0,25

11. Misalnya data tersebut di sensor pada $x = 32$, dan estimasi dari parameter eksponensialnya menjadi $\theta = 25$. Berapa Kolmogorov Smirnov statistic nya?
    a. Kurang dari 0,28
    b. Antara 0,28 dan 0,32
    c. Antara 0,32 dan 0,36
    d. Lebih dari 0,36

12. Hitung Anderson Darling statistic untuk data tersebut (Berdasarkan $\theta = 30$)
    a. Kurang dari 0,4
    b. Antara 0,4 dan 0,6
    c. Antara 0,6 dan 0,8
    d. Lebih dari 0,8

Data berikut berdasarkan data yang disensor sebelah kanan (right censoring) untuk 15 orang yaitu waktu sampai meninggal atau waktu sampai disensor (+) untuk nomor 13 - 15:
2,2,3,3+,4,4,4+,4+,5+,6,6,8,8,9,9+

13. Berapakah nilai estimasi survival $S_{15}(7)$ dengan menggunakan estimasi Kaplan Meier
    a. 0,436
    b. 0,218
    c. 0,725
    d. 0,484

14. Berapakah nilai estimasi hazard rate $\hat{H}_{15}(7)$ dengan menggunakan estimasi Nelson Aalen
    a. 0,436
    b. 0,218
    c. 0,725
    d. 0,484
15. Berapakah nilai estimasi survival $\hat{S}_{15}(7)$ dengan menggunakan estimasi Nelson Aalen
    a. 0,436
    b. 0,218
    c. 0,725
    d. 0,484
16. Sebuah credibility factor parsial untuk variabel random X berdasarkan 100 exposure dari X adalah Z = 0,4. Berapa tambahan exposure yang diperlukan agar credibility factornya dapat ditingkatkan sampai paling tidak 0,5?
    a. 56
    b. 57
    c. 58
    d. 59
17. Jumlah klaim per periode S mempunyai distribusi Compound Poisson. Anda sudah menghitung bahwa untuk mendapatkan credibility penuh, diperlukan sampel sebanyak 2670 klaim, apabila distribusi dari severity konstan. Jika distribusi dari severity adalah lognormal dengan mean 1000 dan variansi 1.500.000 Berapakah jumlahklaim yang diperlukan untuk mendepatkan credibility penuh?
    a. 6675
    b. 6700
    c. 6725
    d. 6750
18. Dua buah kotak berisi masing-masing 10 buah bola yang berukuran dan bentuk sama. Kotak pertama berisi 5 bola merah dan 5 bola putih. Kotak kedua berisi 2 bola merah dan 8 bola putih. Sebuah kotak dipilih secara acak (kemungkinannya sama besar), kemudian sebuah bola diambil dari kotak yang dipilih. Bola yang diambil berwarna merah, berapa besar kemungkinan bola itu diambil dari kotak pertama?
    a. 2/7
    b. 3/7
    c. 4/7
    d. 5/7
19. Berkaitan dengan soal no 18 di atas, apabila bola merah tersebut tidak dikembalikan, kemudian diambil lagi sebuah bola dari kotak yang sama, berapakah kemungkinan bola kedua yang diambil berwarna merah juga?
    a. 6/19
    b. 22/63
    c. 7/20
    d. 13/63

20. Sebuah distribusi Possion mempunyai parameter ⊡, di mana prior distribution nya adalah discrete uniform distribution untuk nilai 1, 2 atau 3. Apabila sebuah observasi diketahui bernilai 1. Berapakah mean dari posterior distributionnya?
    a. 1,5
    b. 1,6
    c. 1,7
    d. 1,8
21. Sebuah kelompok risiko mampunyai distribusi frekuensi per tahun mengikuti distribusi Poisson dengan mean λ. Prior distribution dari λ adalah eksponensial dengan mean 2. Seorang tertanggung diketahui memasukkan 2 buah klaim dalam 1 tahun. Tentukan premi Bayes untuk tertanggung tersebut pada tahun berikutnya.
    a. 1
    b. 3/2
    c. 2
    d. 5/2

Data berikut untuk nomor 22 – 23

Sebuah perusahaan asuransi memiliki dua kelompok polis. Diasumsikan kedua kelompok tersebut memiliki jumlah polis yang sama. Untuk 3 tahun pertama aggregate claim adalah sebagai berikut (dalam milyar rupiah)

Kelompok	Tahun 1	Tahun 2	Tahun 3
1	5	8	11
2	11	13	12

22. Tentukan Buhlmann credibility premium untuk kelompok 1 tahun ke empat.
    a. 8,0
    b. 8,2
    c. 8,4
    d. 8,6
23. Dengan menggunakan metode estimasi credibility weighted average dari μ (dapat juga disebut sebagai metode yang mempertahankan kerugian total), tentukan Buhlmann Credibility premium untuk kelompok 1 tahun keempat
    a. 8,0
    b. 8,2
    c. 8,4
    d. 8,6
24. Sebuah natural cubic spline digunakan untuk mengestimasi h(x) berdasarkan empat titik berikut $x_0 = -2$, $x_1 = -1$, $x_2 = 1$, $x_3 = 2$. Apabila diketahui bahwa $f_1(x) = 1 - 9(x+1) + 4{,}5(X+1)^2$, berapakah $f'(-2) + f'(2)$?
    a. -13,5
    b. -4,5
    c. 0
    d. 4,5

25. Sebuah distribusi Binomial dengan n = 3 dan p = 0,4 disimulasikan dengan metode inverse transform dengan uniform random numbers 0,31 ; 0,71 ; 0,66 ; 0,48 ; 0,19 Berapakah jumlah angka random tersebut yang menghasilkan 2?
    a. 1
    b. 2
    c. 3
    d. 4
26. Sebuah fungsi probability density (pdf) f(x) = 1,5 x^2 untuk -1 <= x <= 1. Simulasi dengan random numbers [0,1] digunakan dan angka randomnya adalah 0,5005 dan 0,2440. Dengan menggunakan inverse transform method, berapakah jumlah dari hasil random tersebut?
    a. -0,7
    b. -0,6
    c. -0,5
    d. -0,4

Informasi berikut digunakan untuk soal 27 – 30. Besaran klaim di bawah ini (dalam juta) di amati dalam sebuah rentang pengamatan: 3, 4, 8, 10, 12, 13, 14, 25, 40, 45.

27. Berapa nilai estimasi dari θ yaitu parameter dari distribusi eksponensial data tersebut dengan menggunakan pendekatan metode momen.
    a. 12,5
    b. 13,9
    c. 17,4
    d. 18,0
28. Berapa nilai estimasi dari θ yaitu parameter dari distribusi eksponensial data tersebut dengan menggunakan pendekatan metode percentile, apabila diambil percentile 50%.
    a. 12,5
    b. 13,9
    c. 17,4
    d. 18,0
29. Apabila perusahaan asuransi menerapakan angka maksimum klaim sebesar 30, dan distribusi eksponensial dengan parameter seperti no 2 di atas, berapakah angka ekspektasi pembayaran klaim dengan limit 30 atau E[X ∧ 30]
    a. 12,5
    b. 14,6
    c. 14,9
    d. 16,5
30. Berapa nilai dari θ untuk parameter dari distribusi inverse eksponensial dengan metode Maximum Likelihood Estimation
    a. 6,5
    b. 8,9
    c. 12,5
    d. 12,9

Tables for Exam C/4

The reading material for Exam C/4 includes a variety of textbooks. Each text has a set of probability distributions that are used in its readings. For those distributions used in more than one text, the choices of parameterization may not be the same in all of the books. This may be of educational value while you study, but could add a layer of uncertainty in the examination. For this latter reason, we have adopted one set of parameterizations to be used in examinations. This set will be based on Appendices A & B of *Loss Models: From Data to Decisions* by Klugman, Panjer and Willmot. A slightly revised version of these appendices is included in this note. A copy of this note will also be distributed to each candidate at the examination.

Each text also has its own system of dedicated notation and terminology. Sometimes these may conflict. If alternative meanings could apply in an examination question, the symbols will be defined.

For Exam C/4, in addition to the abridged table from *Loss Models*, sets of values from the standard normal and chi-square distributions will be available for use in examinations. These are also included in this note.

When using the normal distribution, choose the nearest z-value to find the probability, or if the probability is given, choose the nearest z-value. No interpolation should be used.

> Example: If the given z-value is 0.759, and you need to find $\Pr(Z < 0.759)$ from the normal distribution table, then choose the probability for z-value = 0.76: $\Pr(Z < 0.76) = 0.7764$.

When using the normal approximation to a discrete distribution, use the continuity correction.

The density function for the standard normal distribution is $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

NORMAL DISTRIBUTION TABLE

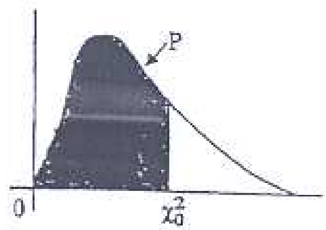
Entries represent the area under the standardized normal distribution from -∞ to z, Pr(Z<z)

The value of z to the first decimal is given in the left column. The second decimal place is given in the top row

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Values of z for selected values of Pr(Z<z)							
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
Pr(Z<z)	0.800	0.850	0.900	0.950	0.975	0.990	0.995

χ^2 Distribution



The table below gives the value x_0^2 for which $P[x^2 < x_0^2] = P$ for a given number of degrees of freedom and a given value of P.

Degrees of Freedom	Values of P									
	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
1	---	---	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.01	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997

Excerpts from the Appendices to *Loss Models: From Data to Decisions, 2nd edition*

June 20, 2007

Appendix A

An Inventory of Continuous Distributions

A.1 Introduction

The incomplete gamma function is given by

$$\Gamma(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\, dt, \quad \alpha > 0,\ x > 0$$

$$\text{with } \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt, \quad \alpha > 0.$$

Also, define

$$G(\alpha; x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt, \quad x > 0.$$

At times we will need this integral for nonpositive values of α. Integration by parts produces the relationship

$$G(\alpha; x) = -\frac{x^\alpha e^{-x}}{\alpha} + \frac{1}{\alpha} G(\alpha + 1; x)$$

This can be repeated until the first argument of G is $\alpha + k$, a positive number. Then it can be evaluated from

$$G(\alpha + k; x) = \Gamma(\alpha + k)[1 - \Gamma(\alpha + k; x)].$$

The incomplete beta function is given by

$$\beta(a, b; x) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1 - t)^{b-1}\, dt, \quad a > 0,\ b > 0,\ 0 < x < 1.$$

A.2 Transformed beta family

A.2.2 Three-parameter distributions

A.2.2.1 Generalized Pareto (beta of the second kind)—α, θ, τ

$$\begin{aligned}
f(x) &= \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\frac{\theta^\alpha x^{\tau-1}}{(x+\theta)^{\alpha+\tau}} \qquad F(x) = \beta(\tau,\alpha;u), \quad u = \frac{x}{x+\theta} \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}, \quad -\tau < k < \alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k\tau(\tau+1)\cdots(\tau+k-1)}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}\beta(\tau+k,\alpha-k;u) + x^k[1-F(x)], \quad k > -\tau \\
\text{mode} &= \theta\frac{\tau-1}{\alpha+1}, \quad \tau > 1, \text{ else } 0
\end{aligned}$$

A.2.2.2 Burr (Burr Type XII, Singh-Maddala)—α, θ, γ

$$\begin{aligned}
f(x) &= \frac{\alpha\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^{\alpha+1}} \qquad F(x) = 1-u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\gamma} \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(1+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)}, \quad -\gamma < k < \alpha\gamma \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(1+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)}\beta(1+k/\gamma,\alpha-k/\gamma;1-u) + x^k u^\alpha, \quad k > -\gamma \\
\text{mode} &= \theta\left(\frac{\gamma-1}{\alpha\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0
\end{aligned}$$

A.2.2.3 Inverse Burr (Dagum)—τ, θ, γ

$$\begin{aligned}
f(x) &= \frac{\tau\gamma(x/\theta)^{\gamma\tau}}{x[1+(x/\theta)^\gamma]^{\tau+1}} \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}, \quad -\tau\gamma < k < \gamma \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}\beta(\tau+k/\gamma,1-k/\gamma;u) + x^k[1-u^\tau], \quad k > -\tau\gamma \\
\text{mode} &= \theta\left(\frac{\tau\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \tau\gamma > 1, \text{ else } 0
\end{aligned}$$

A.2.3 Two-parameter distributions

A.2.3.1 Pareto (Pareto Type II, Lomax)—α, θ

$$f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \qquad F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha$$

$$\mathrm{E}[X^k] = \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha$$

$$\mathrm{E}[X^k] = \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer}$$

$$\mathrm{E}[X \wedge x] = \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1$$

$$\mathrm{E}[X \wedge x] = -\theta \ln\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1$$

$$\mathrm{E}[(X \wedge x)^k] = \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1, \alpha-k; x/(x+\theta)] + x^k\left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{all } k$$

$$\text{mode} = 0$$

A.2.3.2 Inverse Pareto—τ, θ

$$f(x) = \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} \qquad F(x) = \left(\frac{x}{x+\theta}\right)^\tau$$

$$\mathrm{E}[X^k] = \frac{\theta^k\Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1$$

$$\mathrm{E}[X^k] = \frac{\theta^k(-k)!}{(\tau-1)\cdots(\tau+k)}, \quad \text{if } k \text{ is a negative integer}$$

$$\mathrm{E}[(X \wedge x)^k] = \theta^k\tau\int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k}dy + x^k\left[1 - \left(\frac{x}{x+\theta}\right)^\tau\right], \quad k > -\tau$$

$$\text{mode} = \theta\frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0$$

A.2.3.3 Loglogistic (Fisk)—γ, θ

$$f(x) = \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2} \qquad F(x) = u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}$$

$$\mathrm{E}[X^k] = \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma$$

$$\mathrm{E}[(X \wedge x)^k] = \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma$$

$$\text{mode} = \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0$$

A.2.3.4 Paralogistic—α, θ

This is a Burr distribution with $\gamma = \alpha$.

$$
\begin{aligned}
f(x) &= \frac{\alpha^2(x/\theta)^\alpha}{x[1+(x/\theta)^\alpha]^{\alpha+1}} \qquad F(x) = 1 - u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\alpha} \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(1+k/\alpha)\Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)}, \quad -\alpha < k < \alpha^2 \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(1+k/\alpha)\Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)}\beta(1+k/\alpha, \alpha-k/\alpha; 1-u) + x^k u^\alpha, \quad k > -\alpha \\
\text{mode} &= \theta\left(\frac{\alpha-1}{\alpha^2+1}\right)^{1/\alpha}, \quad \alpha > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.5 Inverse paralogistic—τ, θ

This is an inverse Burr distribution with $\gamma = \tau$.

$$
\begin{aligned}
f(x) &= \frac{\tau^2(x/\theta)^{\tau^2}}{x[1+(x/\theta)^\tau]^{\tau+1}} \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\tau}{1+(x/\theta)^\tau} \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k/\tau)\Gamma(1-k/\tau)}{\Gamma(\tau)}, \quad -\tau^2 < k < \tau \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\tau+k/\tau)\Gamma(1-k/\tau)}{\Gamma(\tau)}\beta(\tau+k/\tau, 1-k/\tau; u) + x^k[1-u^\tau], \quad k > -\tau^2 \\
\text{mode} &= \theta(\tau-1)^{1/\tau}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.3 Transformed gamma family

A.3.2 Two-parameter distributions

A.3.2.1 Gamma—α, θ

$$
\begin{aligned}
f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)} \qquad F(x) = \Gamma(\alpha; x/\theta) \\
M(t) &= (1-\theta t)^{-\alpha}, \quad t < 1/\theta \qquad \mathrm{E}[X^k] = \frac{\theta^k\Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad k > -\alpha \\
\mathrm{E}[X^k] &= \theta^k(\alpha+k-1)\cdots\alpha, \quad \text{if } k \text{ is an integer}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\alpha+k)}{\Gamma(\alpha)}\Gamma(\alpha+k; x/\theta) + x^k[1-\Gamma(\alpha; x/\theta)], \quad k > -\alpha \\
&= \alpha(\alpha+1)\cdots(\alpha+k-1)\theta^k\Gamma(\alpha+k; x/\theta) + x^k[1-\Gamma(\alpha; x/\theta)], \quad k \text{ an integer} \\
\text{mode} &= \theta(\alpha-1), \quad \alpha > 1, \text{ else } 0
\end{aligned}
$$

A.3.2.2 Inverse gamma (Vinci)—α, θ

$$\begin{aligned}
f(x) &= \frac{(\theta/x)^\alpha e^{-\theta/x}}{x\Gamma(\alpha)} \qquad F(x) = 1 - \Gamma(\alpha; \theta/x) \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad k<\alpha \qquad \mathrm{E}[X^k] = \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\alpha-k)}{\Gamma(\alpha)}[1-\Gamma(\alpha-k;\theta/x)] + x^k\Gamma(\alpha;\theta/x) \\
&= \frac{\theta^k\Gamma(\alpha-k)}{\Gamma(\alpha)}G(\alpha-k;\theta/x) + x^k\Gamma(\alpha;\theta/x), \text{ all } k \\
\text{mode} &= \theta/(\alpha+1)
\end{aligned}$$

A.3.2.3 Weibull—θ, τ

$$\begin{aligned}
f(x) &= \frac{\tau(x/\theta)^\tau e^{-(x/\theta)^\tau}}{x} \qquad F(x) = 1 - e^{-(x/\theta)^\tau} \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1+k/\tau), \quad k>-\tau \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k\Gamma(1+k/\tau)\Gamma[1+k/\tau;(x/\theta)^\tau] + x^k e^{-(x/\theta)^\tau}, \quad k>-\tau \\
\text{mode} &= \theta\left(\frac{\tau-1}{\tau}\right)^{1/\tau}, \quad \tau>1, \text{ else } 0
\end{aligned}$$

A.3.2.4 Inverse Weibull (log Gompertz)—θ, τ

$$\begin{aligned}
f(x) &= \frac{\tau(\theta/x)^\tau e^{-(\theta/x)^\tau}}{x} \qquad F(x) = e^{-(\theta/x)^\tau} \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1-k/\tau), \quad k<\tau \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k\Gamma(1-k/\tau)\{1-\Gamma[1-k/\tau;(\theta/x)^\tau]\} + x^k\left[1-e^{-(\theta/x)^\tau}\right], \quad \text{all } k \\
&= \theta^k\Gamma(1-k/\tau)G[1-k/\tau;(\theta/x)^\tau] + x^k\left[1-e^{-(\theta/x)^\tau}\right] \\
\text{mode} &= \theta\left(\frac{\tau}{\tau+1}\right)^{1/\tau}
\end{aligned}$$

A.3.3 One-parameter distributions

A.3.3.1 Exponential—θ

$$\begin{aligned}
f(x) &= \frac{e^{-x/\theta}}{\theta} \qquad F(x) = 1 - e^{-x/\theta} \\
M(t) &= (1-\theta t)^{-1} \qquad \mathrm{E}[X^k] = \theta^k\Gamma(k+1), \quad k>-1 \\
\mathrm{E}[X^k] &= \theta^k k!, \quad \text{if } k \text{ is an integer} \\
\mathrm{E}[X\wedge x] &= \theta(1-e^{-x/\theta}) \\
\mathrm{E}[(X\wedge x)^k] &= \theta^k\Gamma(k+1)\Gamma(k+1;x/\theta) + x^k e^{-x/\theta}, \quad k>-1 \\
&= \theta^k k!\Gamma(k+1;x/\theta) + x^k e^{-x/\theta}, \quad k \text{ an integer} \\
\text{mode} &= 0
\end{aligned}$$

A.3.3.2 Inverse exponential—θ

$$\begin{aligned} f(x) &= \frac{\theta e^{-\theta/x}}{x^2} \qquad F(x) = e^{-\theta/x} \\ \mathrm{E}[X^k] &= \theta^k\Gamma(1-k), \quad k<1 \\ \mathrm{E}[(X\wedge x)^k] &= \theta^k G(1-k;\theta/x) + x^k(1-e^{-\theta/x}), \quad \text{all } k \\ \text{mode} &= \theta/2 \end{aligned}$$

A.4 Other distributions

A.4.1.1 Lognormal—μ,σ (μ can be negative)

$$\begin{aligned} f(x) &= \frac{1}{x\sigma\sqrt{2\pi}}\exp(-z^2/2) = \phi(z)/(\sigma x), \quad z = \frac{\ln x - \mu}{\sigma} \qquad F(x) = \Phi(z) \\ \mathrm{E}[X^k] &= \exp(k\mu + k^2\sigma^2/2) \\ \mathrm{E}[(X\wedge x)^k] &= \exp(k\mu + k^2\sigma^2/2)\Phi\left(\frac{\ln x - \mu - k\sigma^2}{\sigma}\right) + x^k[1-F(x)] \\ \text{mode} &= \exp(\mu - \sigma^2) \end{aligned}$$

A.4.1.2 Inverse Gaussian—μ,θ

$$\begin{aligned} f(x) &= \left(\frac{\theta}{2\pi x^3}\right)^{1/2}\exp\left(-\frac{\theta z^2}{2x}\right), \quad z = \frac{x-\mu}{\mu} \\ F(x) &= \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] + \exp\left(\frac{2\theta}{\mu}\right)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \quad y = \frac{x+\mu}{\mu} \\ M(t) &= \exp\left[\frac{\theta}{\mu}\left(1-\sqrt{1-\frac{2t\mu^2}{\theta}}\right)\right], \quad t < \frac{\theta}{2\mu^2}, \qquad \mathrm{E}[X] = \mu, \quad \mathrm{Var}[X] = \mu^3/\theta \\ \mathrm{E}[X\wedge x] &= x - \mu z\Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] - \mu y\exp\left(\frac{2\theta}{\mu}\right)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right] \end{aligned}$$

A.4.1.3 log-t—r,μ,σ (μ *can be negative*)

Let Y have a t distribution with r degrees of freedom. Then $X = \exp(\sigma Y + \mu)$ has the log-t distribution. Positive moments do not exist for this distribution. Just as the t distribution has a heavier tail than the normal distribution, this distribution has a heavier tail than the lognormal distribution.

$$\begin{aligned} f(x) &= \frac{\Gamma\left(\frac{r+1}{2}\right)}{x\sigma\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)\left[1+\frac{1}{r}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right]^{(r+1)/2}}, \\ F(x) &= F_r\left(\frac{\ln x - \mu}{\sigma}\right) \text{ with } F_r(t) \text{ the cdf of a t distribution with } r \text{ d.f.,} \end{aligned}$$

$$F(x) = \begin{cases} \frac{1}{2}\beta\left[\frac{r}{2}, \frac{1}{2}; \frac{r}{r + \left(\frac{\ln x - \mu}{\sigma}\right)^2}\right], & 0 < x \le e^{\mu}, \\ 1 - \frac{1}{2}\beta\left[\frac{r}{2}, \frac{1}{2}; \frac{r}{r + \left(\frac{\ln x - \mu}{\sigma}\right)^2}\right], & x \ge e^{\mu}. \end{cases}$$

A.4.1.4 Single-parameter Pareto—α, θ

$$\begin{aligned} f(x) &= \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}, \quad x > \theta & F(x) &= 1 - (\theta/x)^{\alpha}, \quad x > \theta \\ \mathrm{E}[X^k] &= \frac{\alpha\theta^k}{\alpha - k}, \quad k < \alpha & \mathrm{E}[(X \wedge x)^k] &= \frac{\alpha\theta^k}{\alpha - k} - \frac{k\theta^{\alpha}}{(\alpha - k)x^{\alpha-k}}, \quad x \ge \theta \\ \text{mode} &= \theta \end{aligned}$$

Note: Although there appears to be two parameters, only α is a true parameter. The value of θ must be set in advance.

A.5 Distributions with finite support

For these two distributions, the scale parameter θ is assumed known.

A.5.1.1 Generalized beta—a, b, θ, τ

$$\begin{aligned} f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{\tau}{x}, \quad 0 < x < \theta, \quad u = (x/\theta)^{\tau} \\ F(x) &= \beta(a, b; u) \\ \mathrm{E}[X^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)}, \quad k > -a\tau \\ \mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k/\tau)}{\Gamma(a)\Gamma(a+b+k/\tau)} \beta(a+k/\tau, b; u) + x^k[1 - \beta(a, b; u)] \end{aligned}$$

A.5.1.2 beta—a, b, θ

$$\begin{aligned} f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^a (1-u)^{b-1} \frac{1}{x}, \quad 0 < x < \theta, \quad u = x/\theta \\ F(x) &= \beta(a, b; u) \\ \mathrm{E}[X^k] &= \frac{\theta^k \Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)}, \quad k > -a \\ \mathrm{E}[X^k] &= \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)}, \quad \text{if } k \text{ is an integer} \\ \mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k a(a+1)\cdots(a+k-1)}{(a+b)(a+b+1)\cdots(a+b+k-1)} \beta(a+k, b; u) \\ &\quad + x^k[1 - \beta(a, b; u)] \end{aligned}$$

Appendix B

An Inventory of Discrete Distributions

B.1 Introduction

The 16 models fall into three classes. The divisions are based on the algorithm by which the probabilities are computed. For some of the more familiar distributions these formulas will look different from the ones you may have learned, but they produce the same probabilities. After each name, the parameters are given. All parameters are positive unless otherwise indicated. In all cases, p_k is the probability of observing k losses.

For finding moments, the most convenient form is to give the factorial moments. The jth factorial moment is $\mu_{(j)} = \mathrm{E}[N(N-1)\cdots(N-j+1)]$. We have $\mathrm{E}[N] = \mu_{(1)}$ and $\mathrm{Var}(N) = \mu_{(2)} + \mu_{(1)} - \mu_{(1)}^2$.

The estimators which are presented are not intended to be useful estimators but rather for providing starting values for maximizing the likelihood (or other) function. For determining starting values, the following quantities are used [where n_k is the observed frequency at k (if, for the last entry, n_k represents the number of observations at k or more, assume it was at exactly k) and n is the sample size]:

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{\infty} k n_k, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{\infty} k^2 n_k - \hat{\mu}^2.$$

When the method of moments is used to determine the starting value, a circumflex (e.g., $\hat{\lambda}$) is used. For any other method, a tilde (e.g., $\tilde{\lambda}$) is used. When the starting value formulas do not provide admissible parameter values, a truly crude guess is to set the product of all λ and β parameters equal to the sample mean and set all other parameters equal to 1. If there are two λ and/or β parameters, an easy choice is to set each to the square root of the sample mean.

The last item presented is the probability generating function,

$$P(z) = \mathrm{E}[z^N].$$

B.2 The $(a, b, 0)$ class

B.2.1.1 Poisson—λ

$$\begin{aligned} p_0 &= e^{-\lambda}, \quad a = 0, \quad b = \lambda \qquad p_k = \frac{e^{-\lambda}\lambda^k}{k!} \\ \mathrm{E}[N] &= \lambda, \quad \mathrm{Var}[N] = \lambda \qquad P(z) = e^{\lambda(z-1)} \end{aligned}$$

B.2.1.2 Geometric—β

$$\begin{aligned} p_0 &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0 \qquad p_k = \frac{\beta^k}{(1+\beta)^{k+1}} \\ \mathrm{E}[N] &= \beta, \quad \mathrm{Var}[N] = \beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-1} \end{aligned}$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 Binomial—$q, m, (0 < q < 1, m$ an integer)

$$\begin{aligned} p_0 &= (1-q)^m, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q} \\ p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \ldots, m \\ \mathrm{E}[N] &= mq, \quad \mathrm{Var}[N] = mq(1-q) \qquad P(z) = [1+q(z-1)]^m. \end{aligned}$$

B.2.1.4 Negative binomial—β, r

$$\begin{aligned} p_0 &= (1+\beta)^{-r}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta} \\ p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \\ \mathrm{E}[N] &= r\beta, \quad \mathrm{Var}[N] = r\beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-r}. \end{aligned}$$

B.3 The $(a, b, 1)$ class

To distinguish this class from the $(a, b, 0)$ class, the probabilities are denoted $\Pr(N = k) = p_k^M$ or $\Pr(N = k) = p_k^T$ depending on which subclass is being represented. For this class, p_0^M is arbitrary (that is, it is a parameter) and then p_1^M or p_1^T is a specified function of the parameters a and b. Subsequent probabilities are obtained recursively as in the $(a, b, 0)$ class: $p_k^M = (a + b/k)p_{k-1}^M$, $k = 2, 3, \ldots$, with the same recursion for p_k^T. There are two sub-classes of this class. When discussing their members, we often refer to the "corresponding" member of the $(a, b, 0)$ class. This refers to the member of that class with the same values for a and b. The notation p_k will continue to be used for probabilities for the corresponding $(a, b, 0)$ distribution.

B.3.1 The zero-truncated subclass

The members of this class have $p_0^T = 0$ and therefore it need not be estimated. These distributions should only be used when a value of zero is impossible. The first factorial moment is $\mu_{(1)} = (a+b)/[(1-a)(1-p_0)]$, where p_0 is the value for the corresponding member of the $(a, b, 0)$ class. For the logarithmic distribution (which has no corresponding member), $\mu_{(1)} = \beta/\ln(1+\beta)$. Higher factorial moments are obtained recursively with the same formula as with the $(a, b, 0)$ class. The variance is $(a+b)[1-(a+b+1)p_0]/[(1-a)(1-p_0)]^2$. For those members of the subclass which have corresponding $(a, b, 0)$ distributions, $p_k^T = p_k/(1-p_0)$.

B.3.1.1 Zero-truncated Poisson—λ

$$\begin{aligned}
p_1^T &= \frac{\lambda}{e^\lambda - 1}, \quad a = 0, \quad b = \lambda, \\
p_k^T &= \frac{\lambda^k}{k!(e^\lambda - 1)}, \\
\mathrm{E}[N] &= \lambda/(1 - e^{-\lambda}), \quad \mathrm{Var}[N] = \lambda[1 - (\lambda + 1)e^{-\lambda}]/(1 - e^{-\lambda})^2, \\
\tilde{\lambda} &= \ln(n\hat{\mu}/n_1), \\
P(z) &= \frac{e^{\lambda z} - 1}{e^\lambda - 1}.
\end{aligned}$$

B.3.1.2 Zero-truncated geometric—β

$$\begin{aligned}
p_1^T &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0, \\
p_k^T &= \frac{\beta^{k-1}}{(1+\beta)^k}, \\
\mathrm{E}[N] &= 1 + \beta, \quad \mathrm{Var}[N] = \beta(1+\beta), \\
\hat{\beta} &= \hat{\mu} - 1, \\
P(z) &= \frac{[1 - \beta(z-1)]^{-1} - (1+\beta)^{-1}}{1 - (1+\beta)^{-1}}.
\end{aligned}$$

This is a special case of the zero-truncated negative binomial with $r = 1$

B.3.1.3 Logarithmic—β

$$\begin{aligned}
p_1^T &= \frac{\beta}{(1+\beta)\ln(1+\beta)}, \quad a = \frac{\beta}{1+\beta}, \quad b = -\frac{\beta}{1+\beta}, \\
p_k^T &= \frac{\beta^k}{k(1+\beta)^k \ln(1+\beta)}, \\
\mathrm{E}[N] &= \beta/\ln(1+\beta), \quad \mathrm{Var}[N] = \frac{\beta[1 + \beta - \beta/\ln(1+\beta)]}{\ln(1+\beta)}, \\
\tilde{\beta} &= \frac{n\hat{\mu}}{n_1} - 1 \quad \text{or} \quad \frac{2(\hat{\mu} - 1)}{\hat{\mu}}, \\
P(z) &= 1 - \frac{\ln[1 - \beta(z-1)]}{\ln(1+\beta)}.
\end{aligned}$$

This is a limiting case of the zero-truncated negative binomial as $r \to 0$

B.3.1.4 Zero-truncated binomial—q, m, $(0 < q < 1$, m an integer$)$

$$\begin{aligned}
p_1^T &= \frac{m(1-q)^{m-1}q}{1-(1-q)^m}, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q}, \\
p_k^T &= \frac{\binom{m}{k}q^k(1-q)^{m-k}}{1-(1-q)^m}, \quad k = 1, 2, \ldots, m, \\
\mathrm{E}[N] &= \frac{mq}{1-(1-q)^m}, \\
\mathrm{Var}[N] &= \frac{mq[(1-q)-(1-q+mq)(1-q)^m]}{[1-(1-q)^m]^2}, \\
\tilde{q} &= \frac{\tilde{\mu}}{m}, \\
P(z) &= \frac{[1+q(z-1)]^m-(1-q)^m}{1-(1-q)^m}.
\end{aligned}$$

B.3.1.5 Zero-truncated negative binomial—$\beta, r, (r > -1, r \neq 0)$

$$\begin{aligned}
p_1^T &= \frac{r\beta}{(1+\beta)^{r+1}-(1+\beta)}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta}, \\
p_k^T &= \frac{r(r+1)\cdots(r+k-1)}{k![(1+\beta)^r-1]}\left(\frac{\beta}{1+\beta}\right)^k, \\
\mathrm{E}[N] &= \frac{r\beta}{1-(1+\beta)^{-r}}, \\
Var[N] &= \frac{r\beta[(1+\beta)-(1+\beta+r\beta)(1+\beta)^{-r}]}{[1-(1+\beta)^{-r}]^2}, \\
\tilde{\beta} &= \frac{\tilde{\sigma}^2}{\tilde{\mu}}-1, \quad \tilde{r} = \frac{\tilde{\mu}^2}{\tilde{\sigma}^2-\tilde{\mu}}, \\
P(z) &= \frac{[1-\beta(z-1)]^{-r}-(1+\beta)^{-r}}{1-(1+\beta)^{-r}}.
\end{aligned}$$

This distribution is sometimes called the extended truncated negative binomial distribution because the parameter r can extend below 0.

B.3.2 The zero-modified subclass

A zero-modified distribution is created by starting with a truncated distribution and then placing an arbitrary amount of probability at zero. This probability, p_0^M, is a parameter. The remaining probabilities are adjusted accordingly. Values of p_k^M can be determined from the corresponding zero-truncated distribution as $p_k^M = (1-p_0^M)p_k^T$ or from the corresponding $(a, b, 0)$ distribution as $p_k^M = (1-p_0^M)p_k/(1-p_0)$ The same recursion used for the zero-truncated subclass applies

The mean is $1-p_0^M$ times the mean for the corresponding zero-truncated distribution. The variance is $1-p_0^M$ times the zero-truncated variance plus $p_0^M(1-p_0^M)$ times the square of the zero-truncated mean. The probability generating function is $P^M(z) = p_0^M + (1-p_0^M)P(z)$, where $P(z)$ is the probability generating function for the corresponding zero-truncated distribution.

The maximum likelihood estimator of p_0^M is always the sample relative frequency at 0